Furstenberg boundary of a discrete quantum group

Mehrdad Kalantar (University of Houston)
joint work with P. Kasprzak, A. Skalski and R. Vergnioux

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Motivations/History/Background

In this talk: all (quantum) groups are discrete, all C^*-algebras are unital, and all topological spaces are compact Hausdorff.

Recent applications of boundary actions in C^*-algebras:

1. Amenable embeddings of exact C^*-algebras

2. C^*-simplicity

3. Description & uniqueness of traces

Motivations/History/Background

1. Amenable embeddings of exact C^*-algebras

Recall: a C^*-algebra A is *amenable* if there are u.c.p. maps $\phi_i : A \to M_{n_i}$, $\psi_i : M_{n_i} \to A$ such that $\|\psi_i \circ \phi_i(a) - a\| \to 0$ for all $a \in A$.

A C^*-algebra A is *exact* if there is $B \supset A$ and u.c.p. maps $\phi_i : A \to M_{n_i}$, $\psi_i : M_{n_i} \to B$ such that $\|\psi_i \circ \phi_i(a) - a\| \to 0$ for all $a \in A$.

Obviously: every C^*-subalgebra of an amenable C^*-algebra is exact. The converse is also true!

<u>Problem</u>: given an exact C^*-algebra A, find concrete / canonical / meaningful embeddings $A \hookrightarrow B$ where B is amenable.

Motivations/History/Background

1. Amenable embeddings of exact C^*-algebras

Ozawa's conjecture: given exact C^*-algebra A, $\exists$ nuclear C^*-algebra B s.t. $A \subset B \subset I(A)$, where $I(A)$ is the injective envelope of A, i.e. the minimal injective C^*-algebra that contains A.

Theorem (Ozawa 07): True for $A = C^*_{\mathrm{red}}(\mathbb{F}_n)$.
(with $B = \mathbb{F}_n \ltimes_r C(\partial\mathbb{F}_n)$)

Theorem (K.-Kennedy 17): True for exact $A = C^*_{\mathrm{red}}(\Gamma)$.
(with $B = \Gamma \ltimes_r C(\partial_F\Gamma)$)

(Ozawa's conjecture remains open in general.)

Motivations/History/Background

2. C^*-simplicity

Problem: For which groups Γ, $C^*_{\mathrm{red}}(\Gamma)$ is simple? (equivalently, any rep weakly contained in λ is weakly equivalent to λ).

Theorem (Powers 75): $C^*_{\mathrm{red}}(\mathbb{F}_n)$ is simple.

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(many results ...)

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Theorem (K.-Kennedy 17):
$C^*_{\mathrm{red}}(\Gamma)$ is simple iff $\Gamma \curvearrowright \partial_F \Gamma$ is free.

Motivations/History/Background

3. Description & uniqueness of traces

Problem: Describe traces on $C^*_{\mathrm{red}}(\Gamma)$; in particular, for which groups Γ, $C^*_{\mathrm{red}}(\Gamma)$ has a unique trace?

Theorem (Breuillard-K.-Kennedy-Ozawa 17):
Every trace on $C^*_{\mathrm{red}}(\Gamma)$ is supported on the kernel of $\Gamma \curvearrowright \partial_F\Gamma$; in particular, $C^*_{\mathrm{red}}(\Gamma)$ has a unique trace iff $\Gamma \curvearrowright \partial_F\Gamma$ is faithful.

(Furman 03): $\ker(\Gamma \curvearrowright \partial_F\Gamma)$ is the amenable radical of Γ.

Motivations/History/Background

Goal: extend the notion of boundary actions to the quantum case, apply it to similar problems for discrete quantum groups...

Furstenberg boundary: definition

Definition (Furstenberg):

An action $\Gamma \curvearrowright X$ is a *boundary action* if $\overline{\mathrm{co}(\Gamma\nu)}^{\mathrm{weak}^*} = \mathrm{Prob}(X)$ for every $\nu \in \mathrm{Prob}(X)$;
equivalently, the Poisson transform $\mathcal{P}_\nu : C(X) \to \ell^\infty(\Gamma)$ is isometric for every $\nu \in \mathrm{Prob}(X)$, where $\mathcal{P}_\nu(f)(\gamma) := \int_X f(\gamma x)\, d\nu(x)$.

Proposition (Furstenberg):

There is a (unique) maximal Γ-boundary, which we call the Furstenberg boundary and denote by $\partial_F\Gamma$.

Theorem (K.-Kennedy 17):

$C(\partial_F\Gamma) \cong I_\Gamma(\mathbb{C})$ as Γ-C^*-algebras, where $I_\Gamma(\mathbb{C})$ is the Γ-injective envelope of $\mathbb{C}$, i.e. the (unique) minimal injective object in the category of Γ-C^*-algebras (existence of which was proved by Hamana).

Actions of discrete quantum groups

Below, $\mathbb{\Gamma}$ is a discrete quantum group, its coproduct is denoted by Δ. We will use standard notations.

Definition: A (left) action $\mathbb{\Gamma} \curvearrowright A$ is an injective $\alpha \in \mathrm{Mor}(A, c_0(\mathbb{\Gamma}) \otimes A)$ such that

- $(\mathrm{id} \otimes \alpha) \circ \alpha = (\Delta \otimes \mathrm{id}) \circ \alpha$;
- $(c_0(\mathbb{\Gamma}) \otimes \mathbb{1})\, \alpha(A)$ spans a dense subspace of $c_0(\mathbb{\Gamma}) \otimes A$.

In this case we say A is a $\mathbb{\Gamma}$-C*-algebra.

The convolution actions are defined naturally ...

Actions of discrete quantum groups

Definition: Let $\alpha \in \mathrm{Mor}(A, c_0(\mathbb{\Gamma}) \otimes A)$ be an action of $\mathbb{\Gamma}$ on A. We define the *co-kernel of* α to be the von Neumann algebra

$$N_\alpha := \mathrm{VN}\{\, (\mathrm{id} \otimes \nu)\, \alpha(a) \; : \; \nu \in A^*, a \in A \,\} \subset \ell^\infty(\mathbb{\Gamma})\,.$$

We say that the action α is faithful if $N_\alpha = \ell^\infty(\mathbb{\Gamma})$.

A linear map $\Phi : A \to B$ between $\mathbb{\Gamma}$-C^*-algebras is $\mathbb{\Gamma}$-equivariant if

$$\Phi(a * \mu) = \Phi(a) * \mu \qquad (a \in A \text{ and } \mu \in A^*).$$

The (reduced) crossed product $\mathbb{\Gamma} \ltimes_r A$ is the $\mathbb{C}^*$-algebra generated by $(C^*_{\mathrm{red}}(\mathbb{\Gamma}) \otimes I)\, \alpha(A)$ inside $M(\mathcal{K}(\ell^2(\mathbb{\Gamma})) \otimes A)$.

Boundary actions of discrete quantum groups

Definition: A $\mathbb{\Gamma}$-C*-algebra A is a $\mathbb{\Gamma}$-boundary the set $\{\mu * \nu : \mu \in \ell^1(\mathbb{\Gamma}) \text{ is a state}\}$ is weak* dense in the state space of A, for every state $\nu \in A^*$.

Proposition: For a $\mathbb{\Gamma}$-C*-algebra A the following are equivalent:

1. A is a $\mathbb{\Gamma}$-boundary;
2. every ucp $\mathbb{\Gamma}$-equivariant from A into $\ell^\infty(\mathbb{\Gamma})$ is completely isometric;
3. every ucp $\mathbb{\Gamma}$-equivariant map from A into any $\mathbb{\Gamma}$-C*-algebra B is completely isometric.

The Furstenberg boundary of a discrete quantum group

Theorem: Every discrete quantum group $\mathbb{\Gamma}$ admits a (unique) maximal $\mathbb{\Gamma}$-boundary $C(\partial_F \mathbb{\Gamma})$, in the sense that for any $\mathbb{\Gamma}$-boundary A there is a completely isometric ucp $\mathbb{\Gamma}$-equivariant map $A \to C(\partial_F \mathbb{\Gamma})$.

We call $C(\partial_F \mathbb{\Gamma})$ the (algebra of continuous functions on the) Furstenberg boundary of $\mathbb{\Gamma}$.

The Furstenberg boundary of a discrete quantum group

Quick remarks on the construction:
Similarly to Hamana's construction, we obtain a (unique) minimal injective object in the category of $\ell^1(\mathbb{\Gamma})$-C^*-algebras. We need to work a bit to show that the $\ell^1(\mathbb{\Gamma})$-module action on this injective envelop comes from a genuine $\mathbb{\Gamma}$-action.

Immediate from definition:
Proposition: $\mathbb{\Gamma}$ is amenable iff $C(\partial_F \mathbb{\Gamma})$ is trivial.

Applications: uniqueness of trace

Proposition: Every discrete quantum group $\mathbb{\Gamma}$ admits the largest normal amenable quantum subgroup $R(\mathbb{\Gamma})$, that we call the amenable radical of $\mathbb{\Gamma}$.

Proposition: If the co-kernel N_{F} of the action $\mathbb{\Gamma} \curvearrowright C(\partial_F \mathbb{\Gamma})$ is of quotient type, then $N_{\mathrm{F}} = \ell^\infty(\mathbb{\Gamma}/R(\mathbb{\Gamma}))$.

Definition: A $\mathbb{\Gamma}$-invariant vN subalgebra $M \subset \ell^\infty(\mathbb{\Gamma})$ is relatively amenable if $\exists$ ucp $\mathbb{\Gamma}$-equivariant map $\Psi : \ell^\infty(\mathbb{\Gamma}) \to M$.

Theorem: N_{F} is the unique minimal relatively amenable Baaj-Vaes subalgebra of $\ell^\infty(\mathbb{\Gamma})$. It is contained in every other relatively amenable Baaj-Vaes subalgebra of $\ell^\infty(\mathbb{\Gamma})$.

Applications: uniqueness of trace

Theorem: Suppose $\mathbb{\Gamma}$ has a faithful boundary action $\mathbb{\Gamma} \curvearrowright A$ (equivalently, $\mathbb{\Gamma} \curvearrowright C(\partial_F \mathbb{\Gamma})$ is faithful). If $\mathbb{\Gamma}$ is unimodular then the Haar state is the unique trace on $C^*_{\mathrm{red}}(\mathbb{\Gamma})$. If $\mathbb{\Gamma}$ is not unimodular, then it does not admit any $\mathbb{\Gamma}$-invariant functional, nor any KMS-state for the scaling automorphism group.

A partial converse:
Proposition: If $\mathbb{\Gamma}$ be is unimodular with the unique trace property then the amenable radical of $\mathbb{\Gamma}$ is trivial.

Applications: $\mathbb{C}^*$-simplicity

Theorem: Let $\mathbb{\Gamma}$ be a discrete quantum group. Then the following are equivalent:

1. $C^*_{\mathrm{red}}(\mathbb{\Gamma})$ is simple;
2. $\mathbb{\Gamma} \ltimes_r A$ is simple for every $\mathbb{\Gamma}$-boundary A;
3. $\mathbb{\Gamma} \ltimes_r A$ is simple for some $\mathbb{\Gamma}$-boundary A;
4. $\mathbb{\Gamma} \ltimes_r C(\partial_F \mathbb{\Gamma})$ is simple.

Proposition: If $\mathbb{\Gamma}$ is C^*-simple, then $\mathbb{\Gamma}$ has trivial amenable radical.

Amenable embeddings of exact C^*-algebras

Theorem: Let A be a $\mathbb{\Gamma}$-C^*-algebra. Then A is a $\mathbb{\Gamma}$-boundary iff

$$\mathbb{\Gamma} \ltimes_r A \subset I(C^*_{\mathrm{red}}(\mathbb{\Gamma})).$$

In particular, if $\mathbb{\Gamma} \ltimes_r A$ is amenable, then $C^*_{\mathrm{red}}(\mathbb{\Gamma})$ is exact and Ozawa's conjecture holds in this case.

How to find concrete examples?

Classically, concrete examples of boundary actions are often obtained using hyperbolic-type elements or nice boundary convergence properties. Of course such orbital behavior are not quantum-friendly!!

Theorem: Let $\mathbb{\Gamma}$ be a discrete quantum group and let $\mu \in \ell^1(\mathbb{\Gamma})$ be a state. Suppose A is a unital $\mathbb{\Gamma}$-C*-algebra that admits a unique μ-stationary state ν, and such that the map

$$A \ni a \mapsto (\mathrm{id} \otimes \nu)\alpha(a) \in \ell^\infty(\mathbb{\Gamma})$$

is completely isometric. Then A is a $\mathbb{\Gamma}$-boundary.

Definition: A state $\omega \in A^*$ is μ-stationary if $\mu * \omega = \omega$.

A concrete example and applications

Let $\mathbb{\Gamma} = \mathrm{FO}_Q$ be the free orthogonal discrete quantum group with $Q \in M_N(\mathbb{C})$ such that $Q\bar{Q} = \pm I_N$, $N \geq 3$.

The "Gromov boundary" $\mathcal{B}_\infty$ of $\mathbb{\Gamma}$ was defined by Vaes and Vergnioux, it is a unital $\mathbb{\Gamma}$-C*-algebra that shares many properties of the (continuous functions on the) Gromov boundary of the free group.

Vaes–Vergnioux proved that $\mathcal{B}_\infty$ admits a state ν_∞ such that $\pi_{\nu_\infty}(\mathcal{B}_\infty)''$ is canonically identified with the Poisson boundary of $\mathbb{\Gamma}$.

Concrete example and applications

Theorem: The Poisson state ν_∞ is the unique $\mathrm{qtr}_{M_N(\mathbb{C})}$-stationary state on $\mathcal{B}_\infty$.

Corollary: We have:

1. The Gromov boundary $\mathcal{B}_\infty$ is a $\mathbb{\Gamma}$-boundary.
2. Ozawa's conjecture holds for $C^*_{\mathrm{red}}(\mathbb{\Gamma})$; (Vaes-Vergnioux proved that $\mathbb{\Gamma} \ltimes_r \mathcal{B}_\infty$ is amenable).
3. The crossed product $\mathbb{\Gamma} \ltimes_r \mathcal{B}_\infty$ is simple, provided that $\|Q\|^8 \leq \frac{3}{8}\mathrm{Tr}(QQ^*)$; (Vaes-Vergnioux proved that $C^*_{\mathrm{red}}(\mathbb{\Gamma})$ is simple in this case).

Concrete example and applications

Theorem: The action $\mathbb{\Gamma} \curvearrowright \mathcal{B}_\infty$ is faithful.

Corollary: If Q is unitary, $C^*_{\mathrm{red}}(\mathbb{\Gamma})$ has a unique trace, and otherwise it does not admit any KMS state for the scaling automorphism group.

Thank You!